



THE USE OF FRACTION TILES AS MANIPULATIVE MEDIA IN FRACTION LEARNING TO IMPROVE ALGEBRAIC THINKING SKILLS OF FOURTH GRADE ELEMENTARY SCHOOL STUDENTS

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Abstract

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Algebraic thinking is a fundamental mathematical ability that supports students' understanding of relationships, patterns, and structures, particularly in learning fractions at the elementary school level. This study aimed to examine the effectiveness of using Fraction Tiles as manipulative media in fraction learning to enhance fourth-grade students' algebraic thinking skills. A pre-experimental method with a one-group pretest–posttest design was employed involving 30 elementary school students. The learning activities emphasized concrete manipulation, visual representation, pattern exploration, and symbolic expression of fraction relationships using Fraction Tiles. Students' algebraic thinking abilities were assessed across four indicators: identifying equivalence and relationships among fractions, representing fractions in multiple forms, drawing simple generalizations from fraction patterns, and using mathematical symbols to express relationships. Data were analyzed using descriptive statistics and a Wilcoxon Signed Rank Test. The results revealed a substantial improvement in students' algebraic thinking, as indicated by higher median posttest scores compared to pretest scores. The Wilcoxon Signed Rank Test showed a statistically significant difference ($p = 0.000$), indicating that there was a significant change between pre test and post-test scores. This result confirms that the use of Fraction Tiles as a manipulative medium had a meaningful impact on enhancing students' algebraic thinking abilities. These findings suggest that the use of Fraction Tiles is highly effective in fostering algebraic thinking and provides a meaningful instructional approach for fraction learning in elementary education.

Keywords: Algebraic Thinking, Fraction Learning, Manipulative Media, Elementary School Students

Abstrak

Berpikir aljabar merupakan kemampuan matematis fundamental yang mendukung pemahaman siswa terhadap hubungan, pola, dan struktur, khususnya dalam pembelajaran pecahan di tingkat sekolah dasar. Penelitian ini bertujuan untuk mengkaji efektivitas penggunaan Fraction Tiles sebagai media manipulatif dalam pembelajaran pecahan untuk meningkatkan kemampuan berpikir aljabar siswa kelas IV sekolah dasar. Penelitian ini menggunakan metode pra-eksperimen dengan desain one-group pretest–posttest yang melibatkan 30 siswa sekolah dasar. Kegiatan pembelajaran menekankan pada manipulasi benda konkret, representasi visual, eksplorasi pola, dan ekspresi simbolik dari hubungan antarpecahan menggunakan Fraction Tiles. Kemampuan berpikir aljabar siswa dinilai berdasarkan empat indikator, yaitu mengidentifikasi kesetaraan dan hubungan antarpecahan, merepresentasikan pecahan dalam berbagai bentuk, menarik generalisasi sederhana dari pola pecahan, serta menggunakan simbol matematika untuk menyatakan hubungan. Data dianalisis menggunakan statistik deskriptif dan uji Wilcoxon Signed Rank. Hasil penelitian menunjukkan adanya peningkatan kemampuan berpikir aljabar siswa yang ditunjukkan oleh nilai median posttest yang lebih tinggi dibandingkan dengan nilai median pretest. Hasil uji Wilcoxon Signed Rank menunjukkan adanya perbedaan yang signifikan secara statistik ($p = 0,000$), yang mengindikasikan perubahan signifikan antara skor pretest dan posttest. Temuan ini menunjukkan bahwa penggunaan Fraction Tiles sebagai media manipulatif memberikan dampak positif terhadap peningkatan kemampuan berpikir aljabar siswa. Dengan demikian, penggunaan Fraction Tiles berpotensi menjadi pendekatan pembelajaran yang efektif untuk mengembangkan kemampuan berpikir aljabar melalui pembelajaran pecahan di sekolah dasar..

Kata Kunci: Berpikir Aljabar, Pembelajaran Pecahan, Media Manipulatif, Siswa Sekolah Dasar



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1. Pendahuluan

Mathematics in elementary school has a strategic role as the main foundation in building algebraic thinking skills from an early age. At this level, learning not only aims to master calculation procedures, but also instills conceptual and relational understanding that becomes a bridge between arithmetic and algebra. One of the key materials in the transition process is fractions. However, studies have shown that elementary school students still have significant difficulty understanding fractions as mathematical relationships and structures, rather than just arithmetic operations (Copur-Gencturk, 2021). Fractions are often procedurally perceived and fragmented, thus hindering students' ability to generalize and develop further algebraic reasoning (Pramudiani et al., 2022; Purnomo et al., 2022; Wood et al., 2013).

This difficulty is exacerbated by the characteristics of fractional learning that tend to be abstract and symbolic. The representation of fractions in the form of mathematical notation is often not balanced with the understanding of their relational meaning, leading to misconceptions, both in understanding fractions on a number line and as part of an integer quantity (Kim, 2024). This condition has a direct impact on the inhibition of the development of students' generalization and algebraic reasoning skills, which have been the core of algebraic thinking since elementary education (Sun et al., 2023; Wettergren, 2022). In addition to pedagogical factors, the cognitive aspect of students also contributes to low fractional comprehension. Research shows that working memory skills, fluid intelligence, and mastery of the concept of division are strongly correlated with conceptual understanding of fractions (Li et al., 2025). Students with lower cognitive profiles tend to experience greater barriers, especially when learning only emphasizes procedure without making room for representation and interpretation of concepts (Olfos & Rodríguez, 2019; Stelzer et al., 2023). This emphasizes the importance of a learning approach that focuses on the thought process, not solely on the final outcome.

Various learning strategies have been recommended to overcome these problems, including the use of number lines, the Concrete Representational Abstract (CRA) approach, and realistic context-based learning (Weingarden et al., 2026). Number lines are effective in helping students understand fractions as numbers and relationships, although they require the support of appropriate instructional strategies to enable students to coordinate visual and symbolic representations simultaneously (Kim, 2024). The CRA approach allows students to gradually build fractional concepts from concrete experiences to symbolic abstractions (Flores et al., 2018; Zhang et al., 2022), while context-based and multimedia-based learning has been shown to improve fraction sense by making learning more meaningful and closer to the student experience (Amo-Asante & Bonyah, 2023).

Strengthening the understanding of fractions is closely related to the development of algebraic thinking from an early age. Algebraic thinking at the elementary school level is not interpreted as mastering formal equations, but as the ability to recognize patterns and relationships (Kieran, 2022), represent mathematical situations in various forms, as well as build on simple generalizations of students' arithmetic experiences (Cai & Hwang, 2023). The development of algebraic thinking took place gradually, starting from arithmetic thinking to relational thinking and initial generalizations (Chimoni et al., 2023). Tasks that emphasize functional thinking, relationships between numbers, and pattern investigation play an important role in supporting the transition (R. Ding et al.,

2023; Pinto et al., 2022). In this context, fractional learning contributes significantly to algebraic thinking through the introduction of equivalence patterns, the understanding of interfractional relationships, and the use of mathematical symbols and representations. Understanding fractional equivalence helps students interpret equal signs as symbols of relationships, rather than just markers of outcomes, thus supporting the process of generalization and algebraic reasoning (Pearn et al., 2022). In addition, the understanding of magnitude and the structural relationship between fractions is a strong predictor of the success of algebra at the next level (DeWolf et al., 2015; Gabriel et al., 2023).

The role of fractional learning in developing algebraic thinking becomes more optimal when supported by the use of manipulative media. Manipulative media serves to bridge students' transition from the concrete to the semi-abstract and symbolic, and helps students explore patterns, relationships, and mathematical representations firsthand (M. Ding et al., 2024; Westwood, 2021). One of the relevant manipulative media is Fraction Tiles, which represent fractions visually and tactilely. Fraction Tiles help students understand part-whole relationships, fractional equivalence, and fraction operations and comparisons through active exploration and conceptual visualization (Byrne et al., 2023; Martin et al., 2012). Studies have shown that fraction tiles are effective in strengthening the conceptual understanding of fractions, reducing procedural errors, and helping students meaningfully relate concrete, visual, and symbolic representations (Barbosa & Vale, 2021; Bruce et al., 2023). This approach is in line with the CRA framework and has proven to be beneficial for both regular students and students with learning difficulties (Brave et al., 2025; Vessonen et al., 2021; Westenskow & Moyer-Packenham, 2016). In addition, Fraction Tiles supports exploration-based learning, visualization, and mathematical representation that are essential components in the development of early algebraic thinking.

However, empirical studies that specifically investigate the use of fraction tiles as a means of developing the algebraic thinking skills of elementary school students are still relatively limited. Most previous studies have placed more emphasis on improving cognitive learning outcomes and fractional procedural understanding, and have not explicitly linked them to indicators of algebraic thinking, such as pattern recognition, relationships, representations, and mathematical generalization. In addition, research focusing on grade IV elementary school students as an important transition phase from concrete to semi-abstract thinking is still rare. Based on the research gap, a study is needed that specifically investigates the effect of the use of fraction tiles in fractional learning on the improvement of algebraic thinking skills of grade IV elementary school students. Therefore, this study aims to determine the effect of the use of Fraction Tiles as a manipulative medium in fractional learning on the improvement of algebraic thinking skills of grade IV elementary school students.

2. Metode Penelitian

This study uses a quantitative approach, as it aims to measure the influence of certain treatments on the variables studied objectively through statistical analysis (Creswell, 2014). The research design used is Pre-Experimental Design, which is a research design that involves one group without a comparison group, so that researchers can observe changes before and after the treatment is given (Arifyanto et al., 2025). The design form used is One Group Pretest-Posttest Design, which allows researchers to compare the initial and final conditions of the research subjects after being given treatment (Marsden & Torgerson, 2012). The design of this research is described as follows:

Table 1. Research Design

Pretest	Treatment	Posttest
O_1	X	O_2

Description: O_1 = Initial test of students' algebraic thinking ability
 X = Fraction learning using manipulative media Fraction Tiles

O_2 = Final test of students' algebraic thinking ability

The difference in score between O_1 and O_2 is used to determine the effectiveness of the treatment given.

Research Subjects and Settings

The subjects in this study are grade IV elementary school students. The selection of the subject is based on the consideration of the cognitive development of grade IV students who are at the concrete operational stage towards semi-abstract, so that fractional learning with the help of manipulative media is considered appropriate to develop algebraic thinking skills (Piaget, 2000). The number of students involved in this study was adjusted to the conditions of the classroom and the school where the research was conducted.

Research Instruments

Students' algebraic thinking skills are measured using algebraic thinking tests that include aspects of pattern and relationship recognition, mathematical representation, generalization, and relational reasoning. The indicators measured included students' ability to identify equivalence and relationships between fractions, represent fractions in various forms (concrete, drawing, and symbols), draw simple generalizations of fractional patterns, and use mathematical symbols to express the relationship. The instrument is prepared based on the algebraic thinking construct at the elementary school level which is commonly used in mathematics education research. The validity of the instrument is obtained through expert judgment, while the reliability of the instrument is tested before being used in the research.

Table 2. Algebraic Thinking Ability Grid

Aspects of Algebraic Thinking	Indicator	Question Number
Patterns and Relationships	Identify fractional equivalence patterns	1, 2
Representation	Presenting fractions in various representations (images, symbols)	3, 4
Generalization	Deducing the relationships between fractions based on concrete models	5, 6
Relational Reasoning	Compare two or more fractions based on the part-whole relationship	7, 8
Use of Symbols	Using mathematical symbols to express fractional relationships	9, 10
Total		10 Questions

Data Collection Procedure

The data collection techniques used in this study include: Algebraic thinking ability test, which is given in the form of pretest and posttest. This test is used to measure changes in students' algebraic thinking skills before and after participating in fraction learning using Fraction Tiles (Ercikan & Seixas, 2015). Observation of the learning process, which is used as supporting data to obtain an overview of the implementation of

learning and student involvement during the use of manipulative media Fraction Tiles.

Data Analysis Techniques

Data analysis was carried out through prerequisite tests and hypothesis tests. Pretest and posttest data normality tests use Shapiro–Wilk tests to ensure data are normally distributed before parametric statistical analysis is performed (Ghasemi & Zahediasl, 2012). Furthermore, the Paired Samples t-test was used to determine the difference in the average algebraic thinking ability of students before and after fractional learning using the manipulative media Fraction Tiles (Field, 2018). Hypothesis testing was carried out at a significance level of $\alpha = 0.05$, where the $p < 0.05$ showed a significant influence of the use of Fraction Tiles on students' algebraic thinking ability.

3. Hasil dan Pembahasan

Implementasi Media Fraction Tiles

Media Fraction Tiles is used as a manipulative tool to help students understand the concept of fractions in a concrete and visual way. In learning, students are invited to arrange, compare, and combine pieces of Fraction Tiles that represent parts of a unit. Through this activity, students can directly observe the relationship of the whole section, the equality of fractions, and the comparison between fractions.

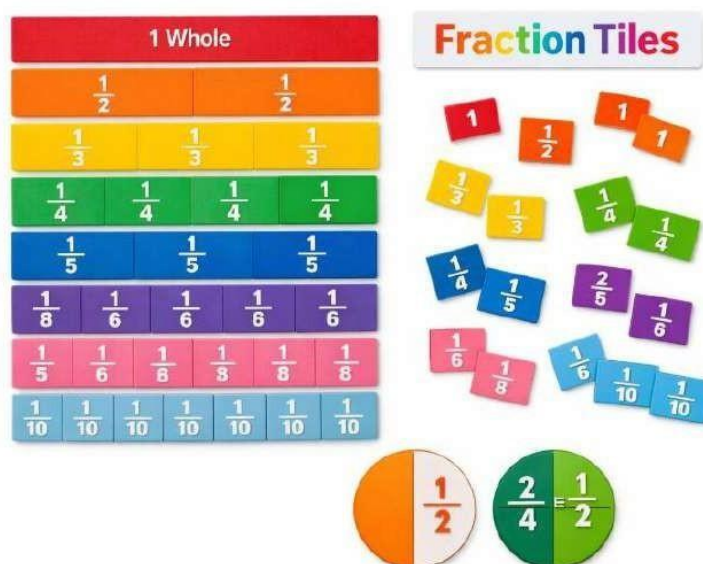


Figure 1. Fraction Tile

The use of Fraction Tiles allows students to explore fraction concepts through hands-on activities, making learning more meaningful and non-abstract. Students can compose multiple fractions to form a single whole number, find patterns of fractional equivalence, as well as represent fractions in various forms before being expressed in mathematical symbols. Thus, this medium helps students develop relational reasoning, generalization skills, and understanding of mathematical structures that are the basis of algebraic thinking. In addition, Fraction Tiles encourages active student involvement in the learning process, both through group discussions and self-exploration. The teacher acts as a facilitator who directs students in asking questions, drawing conclusions, and relating the results of exploration to formal mathematical concepts. The use of this media is expected to improve the understanding of fractional concepts while supporting the development of algebraic thinking skills of elementary school students.

Table 3. Implementation of the Use of Fraction Tiles

Learning Stage	Teacher Activities	Student Activities
Introduction	Convey learning objectives and introduce Fraction Tiles media	Listen to the explanation and get to know the function of Fraction Tiles
Demonstration	Shows how to compile Fraction Tiles into a whole unit and compares fractions	Observing the use of Fraction Tiles
Exploration	Assign exploration tasks using Fraction Tiles and ask lighter questions	Organize, combine, and compare Fraction Tiles pieces
Group Discussions	Guiding discussions and facilitating interaction between students	Discuss, explain the results of the arrangement, and present reasons
Representation	Directs students to represent the results of exploration in the form of images and symbols	Convert the results of manipulations to the shape of mathematical images and symbols
Conclusion	Facilitate students in deducing the concept of fractions and fractional equality	Deducing relationships and fractional patterns
Closing	Provides reinforcement and reflection of learning and outcomes	Reflecting on learning processes

The following descriptive table presents a statistical overview of students' algebraic thinking skills before and after the application of fractional learning using the manipulative media of Fraction Tiles.

Table 4. Descriptive Analysis Results

	N	Red	Median	SD	OR
Pre-Test	30	49.6	49.5	3.53	0.644
Post Test	30	82.0	81.5	3.95	0.721

Based on the table, it can be seen that there is an increase in students' algebraic thinking skills after learning using Fraction Tiles. The average score increased from 49.6 in the pretest to 82.0 in the posttest, which indicates a significant increase. The median score also increased from 49.5 to 81.5, indicating that the improvement occurred evenly in most students. Although the standard deviation of the posttest is slightly larger than that of the

pretest, it still shows a relatively controlled variation in scores. Overall, these descriptive data indicate that the use of Fraction Tiles contributes positively to the improvement of algebraic thinking skills of grade IV elementary school students. The following table presents the results of the normality test of students' algebraic thinking ability data on pretest and posttest scores. The normality test is carried out as part of the analysis prerequisite test to determine the feasibility of using parametric statistical tests at a later stage.

Table 5. Results of Tests of Normality Analysis

			statistic	p
Pre Test	Post Test	Shapiro-Wilk	0.688	<,001
		Kolmogorov-Smirnov	0.280	0.018
		Anderson-Darling	3.57	<,001

Note. Additional results provided by *moretests*

Based on the results of the normality test conducted using three different methods, namely Shapiro-Wilk, Kolmogorov-Smirnov, and Anderson-Darling, it can be concluded that the pre-test and post-test data are not normally distributed. The Shapiro-Wilk test produces a p-value <0.001 for both data, indicating that we reject the null hypothesis that the data are normally distributed. Similarly, the Kolmogorov-Smirnov test produces a p value of 0.018 for both the pre-test and post-test, which also indicates that both data are not normally distributed. The Anderson-Darling test also produces a p-value <0.001, further strengthening the conclusion that the data do not follow a normal distribution. Therefore, because the data do not meet the assumption of normality, the appropriate next step is to use non-parametric analysis, namely the Wilcoxon Signed-Rank Test, to test whether there is a significant difference between the pre-test and post-test.

Table 6. Results of the Hohypothesis Test Analysis

	Null Hypothesis	Test	Sig.	Decision
1	The median of differences between Pre_Test and Post_Test equals 0.	Related-Samples Wilcoxon Signed Rank Test	.000	Reject the null hypothesis.

Asymptotic significances are displayed. The significance level is ,050.

The p-value (significance) obtained from the Wilcoxon test is 0.000, which is less than 0.05 (the commonly used significance threshold). This indicates that we reject the null hypothesis, meaning there is a significant difference between the pre-test and post test scores. In other words, the data indicates a significant change in scores after the intervention (change between pre-test and post-test).

Related-Samples Wilcoxon Signed Rank Test Summary

Total N	30
Test Statistic	465.000
Standard Error	48.597

Standardized Test Statistic	4.784
Asymptotic Sig.(2-sided test)	.000

Based on the results of the Wilcoxon Signed Rank Test, the Asymptotic Sig. (2-sided test) value obtained was 0.000, which is much smaller than the significance limit of 0.05. This indicates that the difference between the pre-test and post-test is statistically significant. Thus, the hypothesis stating that there is a difference between the pre-test and post-test can be accepted. This means that there is a significant change between the two values, which supports the hypothesis that the intervention or change made has a real impact on the measured results.

These results indicate that the use of Fraction Tiles on improving students' algebraic thinking skills is very significant. Therefore, it can be concluded that the use of Fraction Tiles as a manipulative medium in fraction learning is effective in improving fourth-grade elementary school students' algebraic thinking skills. The results of this study reveal that the use of Fraction Tiles as a manipulative medium in fraction learning has a significant impact on improving fourth-grade elementary school students' algebraic thinking skills. This improvement can be analyzed based on the algebraic thinking ability indicators measured in this study.

Ability to Identify Equivalence and Relationships Between Fractions

In this indicator, students show an increase in their ability to recognize the equality relationship between fractions, such as $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$, as well as understand the comparison of values between fractions. Through the use of Fraction Tiles, students can directly compose several fractional pieces to form a single common unit, so that the concept of equality is not only understood symbolically, but also concretely and visually. Theoretically, the understanding of equivalence is the main foundation in the development of algebraic thinking, because students learn to understand mathematical relationships and structures, not just perform calculation procedures (Kieran, 2022). Moreno-Armella et al. (2008), also affirms that the ability to see equivalence and relationships is an early form of elementary school algebra. The findings of this study are in line with the results of the study (Flores et al., 2020; Lee & Lee, 2023) which states that the use of concrete models such as fraction tiles significantly improves the understanding of fractional equality of elementary school students.

Ability to Represent Fractions in Various Forms

The results of the study showed that students became more able to represent fractions in concrete forms (fraction tiles), pictures, and mathematical symbols. In the early stages, students manipulate Fraction Tiles to show a fraction, then describe it, and finally write it down in symbolic form. This process helps students build conceptual understanding gradually.

According to the theory of multiple representations, mathematical understanding will be stronger if students are able to connect different forms of representation (Pope & Choshanov, 2001). In the context of algebraic thinking, the ability to move between representations is an important skill for understanding mathematical structures and relationships (Blanton et al., 2015). Research conducted by Fyfe et al. (2014) It also shows that concrete manipulative-based learning associated with symbolic representation is able to significantly improve students' conceptual understanding.

The Ability to Draw Simple Generalizations of Fractional Patterns

In generalization indicators, students begin to be able to recognize fractional patterns, such as the relationship between numerators and denominators when the size of the part gets smaller or larger. Through the exploration of Fraction Tiles, students can observe

patterns that the larger the denominator, the smaller the fraction part size, and vice versa. From these observations, students are directed to draw general conclusions (generalizations) simply.

Generalization is at the core of algebraic thinking, as students learn to make general statements based on observed patterns and relationships (Mason, 1996). The use of manipulative media supports this process by providing a concrete experience that students can observe and analyze. Research by Carraher et al. (2008) Shows that manipulative and pattern-based activities can help elementary school students develop generalization skills as a basis for algebraic thinking.

The Ability to Use Math Symbols to Express Relationships

The results of the study also showed an increase in students' ability to use mathematical symbols to express the relationship between fractions, such as equals (=) and comparison marks (<). After exploring using Fraction Tiles, students will find it easier to understand the meaning of symbols as representations of relationships, rather than just operational signs. According to Wagner dan Kieran (2018), algebraic thinking emphasizes on the understanding of symbols as a tool for expressing mathematical relationships and structures. Learning that begins with concrete experiences helps students relate the meaning of symbols to the concepts they understand. These findings are in line with research by Rittle-Johnson dan Koedinger (2009), which concludes that the association between concrete and symbolic representations can improve students' relational understanding of mathematical symbols.

4. Conclusion and Suggestions

Based on the results of the research and discussion, it can be concluded that the use of Fraction Tiles as a manipulative medium in fractional learning has a significant effect on improving the algebraic thinking skills of grade IV elementary school students. This is proven by the results of the Paired Samples t-test which shows a significance value of $p < 0.001$, so that H_0 is rejected and H_a is accepted. In addition, the results of the analysis showed that the effect size value (Cohen's $d = -48.0$) was in the very large category, indicating that the use of Fraction Tiles had a strong impact on improving students' algebraic thinking skills. This effect size value confirms that the increase that occurred is not only statistically significant, but also practically meaningful in the context of learning in elementary school. Overall, the use of Fraction Tiles has been shown to be effective in improving students' ability to identify equivalence and relationships between fractions, represent fractions in various forms, draw simple generalizations of fraction patterns, and use mathematical symbols to express mathematical relationships. Therefore, Fraction Tiles can be recommended as an effective learning medium to support the development of algebraic thinking skills of grade IV elementary school students.

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